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Semiclassical expansions in the Toda hierarchy and the Hermitian matrix model*

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Abstract
An iterative algorithm for determining a type of solutions of the dispersionful 2-Toda hierarchy characterized by string equations is developed. This type includes the solution which underlies the large-N limit of the Hermitian matrix model in the one-cut case. It is also shown how the double scaling limit can be naturally formulated in this scheme

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1. Introduction

Since the pioneering works [1, 2] the Toda hierarchy has become one of the paradigmatic examples of the relevance of integrable systems in the theory of random matrix models. As a consequence of the activity in this field a rich theory of the different facets of the Toda hierarchy has been developed [1–10].

The present work is motivated by the applications of the Toda hierarchy theory to the Hermitian matrix model. In this model, the first integrable structure which emerges is the discrete 1-Toda hierarchy [3]

$$\frac{\partial L}{\partial t_j} = [(L^j)_+, L]$$

on semi-infinite tridiagonal matrices

$$L = \Lambda + u_n + v_n \Lambda^T, \qquad n \geqslant 0.$$

Here, Λ is the standard shift matrix and $\mathcal{A}_+$ denotes the upper part (above the main diagonal) of semi-infinite matrices $\mathcal{A}$. This relationship may be conveniently described by considering infinite-dimensional deformations of monic orthogonal polynomials on the real line

$$P_n(z, \boldsymbol{t}) = z^n + \cdots, \qquad \boldsymbol{t} := (t_1, t_2, \ldots), \qquad n \geqslant 0,$$

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with respect to an exponential weight,

$$\int_{-\infty}^{\infty} P_n(z,\boldsymbol{t})P_m(z,\boldsymbol{t})e^{V(z,\boldsymbol{t})}\mathrm{d}z = h_n(\mathbf{t})\delta_{nm}, \qquad V(z,\boldsymbol{t}) := \sum_{k\geqslant 1}(t_k+c_k)z^k,$$

where $\boldsymbol{c} := (c_1, c_2, \ldots)$ is a given set of complex constants. It turns out [3] that the functions

$$\psi_n(z,\boldsymbol{t}) := P_n(z,\boldsymbol{t})\exp\left(\sum_{k\geqslant 1} z^k t_k\right), \qquad n \geqslant 0, \tag{1}$$

satisfy the linear system of the 1-Toda hierarchy on semi-infinite matrices

$$\frac{\partial \psi_n}{\partial t_k} = (L^k)_+\psi_n, \qquad k \geqslant 1, \qquad n \geqslant 0, \tag{2}$$

and have a τ-function representation

$$\psi_n(z,\boldsymbol{t}) = \frac{\tau_n(\boldsymbol{t}-[z^{-1}])}{\tau_n(\boldsymbol{t})} z^n \exp\left(\sum_{k\geqslant 1} z^k t_k\right),$$

provided by the partition function of the Hermitian matrix model

$$\tau_N(\boldsymbol{t}) = Z_N(\boldsymbol{t}) := \int_{\mathbb{R}^N}\prod_{k=1}^{N}(\mathrm{d}x_k e^{V(x_k,\boldsymbol{t})})(\Delta(x_1,\ldots,x_N))^2, \tag{3}$$

where $\Delta(x_1,\ldots,x_n) := \prod_{i>j}(x_i - x_j)$.

The main tools provided by the Toda hierarchy to study the Hermitian matrix model are the linear system (2) and the equations

$$L\psi_n = z\psi_n, \qquad \frac{\partial \psi_n}{\partial z} = M\psi_n, \tag{4}$$

where L and M are the Lax and Orlov operators, respectively, given by

$$L = \Lambda + u_n + v_n\Lambda^T, \qquad M = \sum_{k\geqslant 1} k(t_k+c_k)(\mathcal{L}^{k-1})_+. \tag{5}$$

The same objects can be reformulated in terms of pseudo-differential operators [10] and applied to the study of multi-matrix models.

Many exciting properties of the Hermitian matrix model emerge in its large-N limit. They can be described by taking the continuum limit [1, 9] of the scheme provided by the basic equations (2) and (4). The aim of the present paper is to present an alternative way to analyze the large-N limit. Its main ingredients are:

(1) The use of the *dispersionful* 2-Toda hierarchy: an interpolated continuous version of the 2-Toda hierarchy on infinite matrices introduced by Takasaki and Takebe [12].
(2) An iterative method for determining a class of solutions of the 1-Toda reduction of the dispersionful 2-Toda hierarchy characterized by string equations.

Recent works [14–16] proved that the dispersionful 1-Toda hierarchy verifies the so-called *quasi-triviality* property. It ensures that its solutions are determined by their dispersionless (semiclassical) limits. This is an essential property in our approach as we are able to characterize a solution such that its dispersionless limit describes the planar limit of the Hermitian model. This solution is completely characterized from the iterative scheme without any further continuum limit operation, since that limit is already encoded in the structure of the dispersionful 1-Toda hierarchy.

To motivate our approach, we observe that the large-N limit deals with the asymptotic properties of the partition function

$$Z_N(N\mathbf{t}) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} (\mathrm{d}x_k e^{NV(x_k,\mathbf{t})}) (\Delta(x_1, \ldots, x_N))^2. \tag{6}$$

Here a small parameter $\epsilon := 1/N$, rescaled variables $\mathbf{t} := \epsilon \boldsymbol{t}$ and constants $\mathbf{c} := \epsilon \boldsymbol{c}$ have been introduced. Since ϵ plays the role of the Planck constant $\hbar$, expansions in powers of ϵ are referred to as *semiclassical* expansions. The point is that the same slow variables $\mathbf{t} = \epsilon \boldsymbol{t}$ together with a continuous variable $x := \epsilon n$ are introduced to pass from the standard Toda hierarchy to its dispersionful version [12]. In this way, and due to the fact that $\tau_n(\boldsymbol{t}) = Z_n(\boldsymbol{t})$ is a τ-function of the semi-infinite discrete Toda hierarchy, it is natural to expect that a τ-function $\tau(\epsilon, x, \mathbf{t})$ of the dispersionful 1-Toda hierarchy verifying

$$\tau(\epsilon, \epsilon n, \mathbf{t}) = Z_n(N\mathbf{t}), \tag{7}$$

should describe the large-N limit of the Hermitian model.

Our work includes a scheme for obtaining solutions of the dispersionful 2-Toda hierarchy satisfying the system of *string equations*

$$\mathcal{L} = \bar{\mathcal{L}}, \qquad \mathcal{M} + F(\mathcal{L}) = \overline{\mathcal{M}} + \overline{F}(\bar{\mathcal{L}}), \tag{8}$$

where now $(\mathcal{L}, \mathcal{M})$ and $(\bar{\mathcal{L}}, \overline{\mathcal{M}})$ denote the two pairs of Lax–Orlov operators of the dispersionful 2-Toda hierarchy and $(F, \overline{F})$ are two arbitrary functions.

The first string equation represents the 1-Toda reduction condition and is satisfied by Lax operators of the form

$$\mathcal{L} = \bar{\mathcal{L}} = \Lambda + u + v\Lambda^{-1}, \tag{9}$$

where now $\Lambda := \exp(\epsilon \partial_x)$, and (u, v) are characterized by semiclassical expansions

$$u = \sum_{k\geqslant 0} \epsilon^k u^{(k)}, \qquad v = \sum_{k\geqslant 0} \epsilon^{2k} v^{(2k)}. \tag{10}$$

For $x = 1$ and $\overline{F} \equiv 0$ the constraints (8) lead to solutions $(\mathcal{L}, \mathcal{M})$ which interpolate the finite-difference operators (5), so that the corresponding solution of the dispersionful 1-Toda hierarchy is a candidate to the solution underlying the large-N limit of the Hermitian model. That it is the only possible candidate can be argued from the quasi-triviality property of the dispersionful 1-Toda hierarchy and the fact that, as it is shown in this paper, the terms $(u^{(0)}, v^{(0)})$ of the solution of (8) coincide with those characterizing the leading order in the large N-expansion (planar limit) of the Hermitian model.

Our strategy is inspired by the previous results [17–20] on solution methods for dispersionless string equations. We also develop some useful standard technology of the theory of Lax equations [21–23] in the context of the dispersionful Toda hierarchy. Thus we introduce two generating functions $\mathbb{R}$ and $\mathbb{T}$ related to the resolvent of the Lax operator which play a crucial role in our analysis.

The paper is organized as follows. In the following section, the basic theory of the dispersionful 2-Toda hierarchy and the method of string equations are discussed. In section 3, we deal with the dispersionful 1-Toda hierarchy and its relationship with the Hermitian matrix model from the point of view of the string equations (8). The generating functions $\mathbb{R}$ and $\mathbb{T}$ are introduced and characterized by two important identities. Our main results are derived in section 4 where a scheme for solving the string equations (8) in terms of semiclassical expansions is provided. In particular, we prove that the leading terms of these expansions characterize the planar limit of the Hermitian matrix model. In section 5, it is showed how the double scaling limit method can be naturally implemented in our scheme. Finally in section 6, we summarize the main results and discuss several possible applications and open problems.

2. String equations in the dispersionful 2-Toda hierarchy

2.1. The dispersionful 2-Toda hierarchy

The formulation of the dispersionful 2-Toda hierarchy [12] uses operators of the form

$$\mathcal{A} = \sum_{j\in\mathbb{Z}} a_j(\epsilon, x, \mathbf{t}, \bar{\mathbf{t}})\Lambda^j, \qquad \Lambda := \exp(\epsilon\partial_x), \tag{11}$$

where x is a complex variable and the coefficients are in turn series in the small parameter ϵ

$$a_j(\epsilon, x, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{k\in\mathbb{Z}} \epsilon^k a_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}).$$

Here, $\mathbf{t} = (t_1, t_2, \dots)$ and $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \dots)$ denote two infinite sets of complex variables. The order in ϵ of $\mathcal{A}$ is defined by

$$\mathrm{ord}_\epsilon(\mathcal{A}) := \max\left\{-k \middle| a_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}) \neq 0\right\}.$$

For example $\mathrm{ord}_\epsilon(\epsilon) = -1$ and $\mathrm{ord}_\epsilon(\Lambda) = 0$. In particular, zero-order operators are those with regular coefficients a_j as $\epsilon \to 0$. As usual the $\mathcal{A}_\pm$ parts of $\mathcal{A}$ will denote the truncations of Λ-series in the positive and strictly negative power terms, respectively. Given a function w depending on x, the following notation convention will be henceforth used:

$$w_{[r]} := \Lambda^r w = w(x + r\epsilon), \qquad r \in \mathbb{Z}.$$

The dispersionful 2-Toda hierarchy can be formulated in terms of a pair of formal wavefunctions of the form

$$\begin{aligned} \Psi &= \exp\left(\frac{1}{\epsilon}\mathbb{S}\right), & \mathbb{S} &= \sum_{j=1}^{\infty} t_j z^j + x\log z - \sum_{j\geqslant 1}\frac{1}{jz^j}S_{j+1}, \\ \bar{\Psi} &= z^{-1}\exp\left(\frac{1}{\epsilon}\bar{\mathbb{S}}\right), & \bar{\mathbb{S}} &= \sum_{j=1}^{\infty} \bar{t}_j z^j - x\log z - \bar{S}_0 - \sum_{j\geqslant 1}\frac{1}{jz^j}\bar{S}_{j+1}, \end{aligned} \tag{12}$$

where

$$S_j = \sum_{k\geqslant 0}\epsilon^k S_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}), \qquad \bar{S}_j = \sum_{k\geqslant 0}\epsilon^k \bar{S}_j^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}),$$

These functions $\boldsymbol{\Psi} = \Psi, \overline{\Psi}$ are assumed to satisfy the linear system

$$\epsilon\frac{\partial\boldsymbol{\Psi}}{\partial t_j} = (\mathcal{L}^j)_+\boldsymbol{\Psi}, \qquad \epsilon\frac{\partial\boldsymbol{\Psi}}{\partial\bar{t}_j} = (\bar{\mathcal{L}}^j)_-\boldsymbol{\Psi}, \tag{13}$$

where the Lax operators $\mathcal{L}$ and $\bar{\mathcal{L}}$ are determined by the equations

$$\mathcal{L}\Psi = z\Psi, \qquad \bar{\mathcal{L}}\overline{\Psi} = z\overline{\Psi}, \tag{14}$$

and are assumed [12] to be of zero order in ϵ. We will also use the Orlov operators $\mathcal{M}$ and $\overline{\mathcal{M}}$ characterized by

$$\mathcal{M}\Psi = \epsilon\frac{\partial\Psi}{\partial z}; \qquad \overline{\mathcal{M}\Psi} = \epsilon\frac{\partial\overline{\Psi}}{\partial z}, \tag{15}$$

which satisfy

$$[\mathcal{L}, \mathcal{M}] = [\overline{\mathcal{L}}, \overline{\mathcal{M}}] = \epsilon.$$

Using (12) and (14)–(15) one sees that the following expansions follow:

$$\mathcal{L} = \Lambda + u_0 + u_1\Lambda^{-1} + \cdots, \qquad \mathcal{M} = \sum_{j=1}^{\infty} j t_j \mathcal{L}^{j-1} + x\mathcal{L}^{-1} + \sum_{j\geqslant 1} S_{j+1}\mathcal{L}^{-j-1}$$
$$\bar{\mathcal{L}} = \bar{u}_{-1}\Lambda^{-1} + \bar{u}_0 + \bar{u}_1\Lambda + \cdots, \qquad \overline{\mathcal{M}} = \sum_{j=1}^{\infty} j\bar{t}_j\bar{\mathcal{L}}^{j-1} - (x+\epsilon)\bar{\mathcal{L}}^{-1} + \sum_{j\geqslant 1} \bar{S}_{j+1}\bar{\mathcal{L}}^{-j-1}. \tag{16}$$

Furthermore, (13) can be rewritten in Lax form as

$$\epsilon\frac{\partial K}{\partial t_j} = [(\mathcal{L}^j)_+, K], \qquad \epsilon\frac{\partial K}{\partial \bar{t}_j} = [(\bar{\mathcal{L}}^j)_-, K], \tag{17}$$

where $K = \mathcal{L}, \mathcal{M}, \bar{\mathcal{L}}, \overline{\mathcal{M}}$.

There is also a τ-function representation of the wavefunctions [12]

$$\Psi = \exp\left(\frac{1}{\epsilon}\left(\sum_{j=1}^{\infty} t_j z^j + x\log z\right)\right)\frac{\tau(\epsilon, x, \mathbf{t} - \epsilon[z^{-1}], \bar{\mathbf{t}})}{\tau(\epsilon, x, \bar{\mathbf{t}})},$$
$$\bar{\Psi} = z^{-1}\exp\left(\frac{1}{\epsilon}\left(\sum_{j=1}^{\infty} \bar{t}_j z^j - x\log z\right)\right)\frac{\tau(\epsilon, x+\epsilon, \mathbf{t}, \bar{\mathbf{t}} - \epsilon[z^{-1}])}{\tau(\epsilon, x, \mathbf{t}, \bar{\mathbf{t}})}, \tag{18}$$

where $[z^{-1}] := (1/z, 1/2z^2, 1/3z^3, \ldots)$ and τ is of the form.

$$\tau = \exp\left(\frac{1}{\epsilon^2}\mathbb{F}\right), \qquad \mathbb{F} = \sum_{k\geqslant 0}\epsilon^k F^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}). \tag{19}$$

The dispersionful 2-Toda hierachy arises as a continuum limit of the standard 2-Toda hierachy [11] in which the standard discrete variable n is substituted by a continuous variable x and two sets of *fast* continuous variables $\boldsymbol{t} := \epsilon^{-1}\mathbf{t}, \bar{\boldsymbol{t}} := \epsilon^{-1}\bar{\mathbf{t}}$ are introduced. Thus, the τ-functions of both hierarchies are related by

$$\tau(\epsilon, \epsilon n, \epsilon\boldsymbol{t}, \epsilon\bar{\boldsymbol{t}}) = \tau_n(\boldsymbol{t}, \bar{\boldsymbol{t}}). \tag{20}$$

Our subsequent analysis uses an important result proved by Takasaki and Takebe (proposition 2.7.11. in [12])

Theorem 1. *Suppose that*

$$\mathcal{P}(\epsilon, x\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} p_k(\epsilon, x\Lambda^{-1})\Lambda^k, \qquad \mathcal{Q}(\epsilon, x\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} q_k(\epsilon, x\Lambda^{-1})\Lambda^k,$$
$$\overline{\mathcal{P}}(\epsilon, x\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} \bar{p}_k(\epsilon, x\Lambda^{-1})\Lambda^k, \qquad \overline{\mathcal{Q}}(\epsilon, x\Lambda^{-1}, \Lambda) = \sum_{k\in\mathbb{Z}} \bar{q}_k(\epsilon, x\Lambda^{-1})\Lambda^k,$$

are operators of zero order in ϵ verifying

$$[\mathcal{P}, \mathcal{Q}] = [\overline{\mathcal{P}}, \overline{\mathcal{Q}}] = \epsilon.$$

If $(\mathcal{L}, \mathcal{M})$ and $(\bar{\mathcal{L}}, \overline{\mathcal{M}})$ are operators of zero order in ϵ of the form (16) which satisfy the pair of constraints

$$\mathcal{P}(\epsilon, \mathcal{M}, \mathcal{L}) = \overline{\mathcal{P}}(\epsilon, \overline{\mathcal{M}}, \bar{\mathcal{L}}), \qquad \mathcal{Q}(\epsilon, \mathcal{M}, \mathcal{L}) = \overline{\mathcal{Q}}(\epsilon, \overline{\mathcal{M}}, \bar{\mathcal{L}}), \tag{21}$$

and

$$[\mathcal{L}, \mathcal{M}] = [\bar{\mathcal{L}}, \overline{\mathcal{M}}] = \epsilon, \tag{22}$$

then $(\mathcal{L}, \mathcal{M})$ *and* $(\bar{\mathcal{L}}, \overline{\mathcal{M}})$ *are solutions of the Lax equations (17) of the dispersiondul 2-Toda hierarchy.*

Constraints of the form (21) are called *string equations*. In this work, we are interested in the particular example given by

$$\begin{cases} \mathcal{L} = \bar{\mathcal{L}}, \\ \mathcal{M} + F(\mathcal{L}) = \overline{\mathcal{M}} + \overline{F}(\bar{\mathcal{L}}), \end{cases} \tag{23}$$

where $F(\mathcal{L})$ and $\overline{F}(\bar{\mathcal{L}})$ are arbitrary functions of the form

$$F(\mathcal{L}) := \sum_{j\geqslant 1} j c_j \mathcal{L}^{j-1}, \qquad \overline{F}(\bar{\mathcal{L}}) := \sum_{j\geqslant 1} j \bar{c}_j \bar{\mathcal{L}}^{j-1}.$$

3. The dispersionful 1-Toda hierarchy and the Hermitian model

The first string equation in (23) represents the so-called *tridiagonal* (1-Toda) reduction of the dispersionful 2-Toda hierarchy and implies the following form of the Lax operators:

$$\mathcal{L} = \bar{\mathcal{L}} = \Lambda + u + v\Lambda^{-1}. \tag{24}$$

Thus, as a consequence of the Lax equations, u and v depend on $(\mathbf{t}, \bar{\mathbf{t}})$ through the combination $\mathbf{t} - \bar{\mathbf{t}}$. Moreover (24) implies

$$(\Lambda + u + v\Lambda^{-1})\Psi = z\Psi, \qquad (\Lambda + u + v\Lambda^{-1})\overline{\Psi} = z\overline{\Psi}, \tag{25}$$

so that

$$u = \epsilon^{-1}(S_{2[1]} - S_2), \qquad \log v = \epsilon^{-1}(\bar{S}_{0[-1]} - \bar{S}_0). \tag{26}$$

In order to solve the string equations (23) it is required to characterize the action of the operators $(\mathcal{L}^j)_+$ and $(\bar{\mathcal{L}}^j)_-$ on the wavefunctions Ψ and $\overline{\Psi}$. This calculation is also needed to determine the integrable systems of the dispersionful 1-Toda hierarchy. We start by introducing the two series in z

$$p(z) = z - u + \mathcal{O}\left(\frac{1}{z}\right), \qquad \bar{p}(z) = \frac{v_{[1]}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad z \to \infty, \tag{27}$$

satisfying

$$\Lambda\Psi = p(z)\Psi, \qquad \Lambda\overline{\Psi} = \bar{p}(z)\overline{\Psi}, \tag{28}$$

which according to (25) are determined by

$$\boldsymbol{p}(z) + u + \frac{v}{\boldsymbol{p}_{[-1]}(z)} = z, \tag{29}$$

where $\boldsymbol{p} = p, \bar{p}$. By using (25) it is clear that there are functions $\alpha_j, \beta_j, \bar{\alpha}_j, \bar{\beta}_j$, which depend polynomially in z, such that

$$\begin{aligned} \epsilon\frac{\partial\boldsymbol{\Psi}}{\partial t_j} &= (\mathcal{L}^j)_+\boldsymbol{\Psi} = \alpha_j\boldsymbol{\Psi} + \beta_j\Lambda\boldsymbol{\Psi} = (\alpha_j + \beta_j\boldsymbol{p})\boldsymbol{\Psi}, \\ \epsilon\frac{\partial\boldsymbol{\Psi}}{\partial \bar{t}_j} &= (\bar{\mathcal{L}}^j)_-\boldsymbol{\Psi} = \bar{\alpha}_j\boldsymbol{\Psi} + \bar{\beta}_j\Lambda\boldsymbol{\Psi} = (\bar{\alpha}_j + \bar{\beta}_j\boldsymbol{p})\boldsymbol{\Psi} \end{aligned} \tag{30}$$

and

$$\bar{\alpha}_j = z^j - \alpha_j, \qquad \bar{\beta}_j = -\beta_j. \tag{31}$$

Hence, we have

$$\alpha_j + \beta_j p = \partial_{t_j}\mathbb{S}(z) = z^j + \mathcal{O}\left(\frac{1}{z}\right), \qquad \alpha_j + \beta_j \bar{p} = \partial_{t_j}\bar{\mathbb{S}}(z) = -\partial_{t_j}\bar{S}_0 + \mathcal{O}\left(\frac{1}{z}\right), \tag{32}$$

so that

$$\alpha_j = \frac{1}{2}\big(z^j - \partial_{t_j}\bar{S}_0 - (\beta_j(p+\bar{p}))_{\oplus}\big), \qquad \beta_j = \left(\frac{z^j}{p-\bar{p}}\right)_{\oplus}, \tag{33}$$

where $(\,)_{\oplus}$ and $(\,)_{\ominus}$ stand for the projections of z-series on the subspaces generated by the positive and strictly negative powers, respectively.

At this point it is useful to introduce the generating functions

$$\mathbb{R} := \frac{z}{p-\bar{p}} = \sum_{k\geqslant 0}\frac{R_k(u,v)}{z^k}, \qquad \mathbb{T} := \frac{p+\bar{p}}{p-\bar{p}} = \sum_{k\geqslant 0}\frac{T_k(u,v)}{z^k}, \qquad R_0 = T_0 = 1. \tag{34}$$

By substituting p and $\bar{p}$ by their expressions in terms of $\mathbb{R}$ and $\mathbb{T}$ in the identities

$$u = z + \frac{pp_{[-1]} - \bar{p}\bar{p}_{[-1]}}{\bar{p}_{[-1]} - p_{[-1]}}, \qquad v = \frac{\bar{p}-p}{\bar{p}_{[-1]} - p_{[-1]}}\bar{p}_{[-1]}p_{[-1]}, \tag{35}$$

we obtain the following relations:

$$\begin{cases} \mathbb{T}_{[1]} + \mathbb{T} + \dfrac{2}{z}(u_{[1]} - z)\mathbb{R}_{[1]} = 0, \\ \mathbb{T}^2 - \dfrac{4}{z^2}v_{[1]}\mathbb{R}\mathbb{R}_{[1]} = 1, \end{cases} \tag{36}$$

which allow us to compute recursively the coefficients of the series (34) as polynomials in u, v and their x-translations $u_{[r]}$ and $v_{[r]}$. Indeed, the system (36) implies

$$\begin{cases} 2T_{k+1} = -\displaystyle\sum_{i+j=k+1;\,i,j\geqslant 1} T_i T_j + 4v_{[1]}\sum_{i+j=k-1} R_i R_{j[1]}, \\ R_{k+1} = uR_k + \dfrac{1}{2}[T_{k+1} + T_{k+1[-1]}],\,. \end{cases} \tag{37}$$

For example, the first few coefficients are:

$$T_1 = 0, \qquad R_1 = u, \qquad T_2 = 2v_{[1]}, \qquad R_2 = u^2 + v_{[1]} + v,$$
$$T_3 = 2v_{[1]}(u + u_{[1]}), \qquad R_3 = u^3 + 2uv_{[1]} + 2uv + u_{[1]}v_{[1]} + u_{[-1]}v,$$
$$T_4 = 2v_{[1]}\big(u_{[1]}^2 + uu_{[1]} + u^2 + v_{[2]} + v_{[1]} + v\big).$$

In this way, by taking into account the second equation of (37), one finds

$$\begin{aligned} \partial_{t_j}\mathbb{S}(z) = \alpha_j + \beta_j p &= z^j - \frac{1}{2}\partial_{t_j}\bar{S}_0 - \frac{z}{2\mathbb{R}}(z^{j-1}\mathbb{R})_{\ominus} + \left(\frac{z}{2\mathbb{R}}\mathbb{T}(z^{j-1}\mathbb{R})_{\oplus}\right)_{\ominus} \\ &= z^j - \frac{1}{2}(\partial_{t_j}\bar{S}_0 + R_j) - \frac{1}{2z}T_{j+1[-1]} + \mathcal{O}\left(\frac{1}{z^2}\right), \end{aligned} \tag{38}$$

so that

$$\partial_{t_j}\bar{S}_0 = -R_j, \qquad \partial_{t_j}S_2 = \tfrac{1}{2}T_{j+1[-1]}$$

and then from (26) we get that the flows of the dispersionful 1-Toda hierarchy can be expressed as

$$\epsilon\partial_{t_j}u = \tfrac{1}{2}(T_{j+1} - T_{j+1[-1]}), \qquad \epsilon\partial_{t_j}v = v(R_j - R_{j[-1]}). \tag{39}$$

Furthermore, our calculation implies the following useful relations:

$$(\mathcal{L}^j)_-\Psi = \left(-\frac{1}{2}R_j + \frac{z}{2\mathbb{R}}(z^{j-1}\mathbb{R})_\ominus - \left(\frac{z}{2\mathbb{R}}\mathbb{T}(z^{j-1}\mathbb{R})_\oplus\right)_\ominus\right)\Psi,$$
$$(\mathcal{L}^j)_+\overline{\Psi} = \left(\frac{1}{2}R_j + \frac{z}{2\mathbb{R}}(z^{j-1}\mathbb{R})_\ominus + \left(\frac{z}{2\mathbb{R}}\mathbb{T}(z^{j-1}\mathbb{R})_\oplus\right)_\ominus\right)\overline{\Psi}, \tag{40}$$

for $j \geqslant 1$. In particular, by taking the second equation of (37) into account one finds that as $z \to \infty$

$$(\mathcal{L}^j)_-\Psi = \left(\frac{1}{2z}T_{j+1[-1]} + \mathcal{O}\left(\frac{1}{z^2}\right)\right)\Psi,$$
$$(\mathcal{L}^j)_+\overline{\Psi} = \left(R_j + \frac{1}{2z}T_{j+1} + \mathcal{O}\left(\frac{1}{z^2}\right)\right)\overline{\Psi}. \tag{41}$$

We observe that since $R_0 = 1$, $T_1 = 0$ these last equations hold for $j \geqslant 0$.

By following the analysis of [23] it can be seen that $\mathbb{R}$ and $\mathbb{T}$ are closely related to the resolvent of the Lax operator $\mathcal{L}$,

$$\mathcal{R} := (z - \mathcal{L})^{-1}.$$

Thus, from lemmas 3.5 and 3.18 of [23] one proves that

$$\left(z - \frac{2\mathbb{R}}{(1+\mathbb{T})}\Lambda\right)\mathcal{R}_+ = \mathbb{R}, \qquad \operatorname{Res}\mathcal{R}_+ = \frac{\mathbb{R}}{z},$$

where $\operatorname{Res}\left(\sum c_k\Lambda^k\right) := c_0$.

τ-function representation. It follows from (18) and (25) that the functions u and v can be written in terms of the τ-function as

$$u = \epsilon\frac{\partial}{\partial t_1}\log\frac{\tau(\epsilon, x+\epsilon, \mathbf{t})}{\tau(\epsilon, x, \mathbf{t})}, \qquad v = \frac{\tau(\epsilon, x+\epsilon, \mathbf{t})\tau(\epsilon, x-\epsilon, \mathbf{t})}{\tau^2(\epsilon, x, \mathbf{t})}, \tag{42}$$

where we have set $\mathbf{t} - \bar{\mathbf{t}} \to \mathbf{t}$. On the other hand, it can be proved [15, 16] that the ϵ-expansion of the τ-functions of the dispersionful 1-Toda hierarchy is of the form

$$\tau = \exp\left(\frac{1}{\epsilon^2}\mathbb{F}\right), \qquad \mathbb{F} = \sum_{k\geqslant 0}\epsilon^{2k}F^{(2k)}. \tag{43}$$

As a consequence u and v can be expanded as

$$u = \sum_{k\geqslant 0}\epsilon^k u^{(k)}, \qquad v = \sum_{k\geqslant 0}\epsilon^{2k}v^{(2k)}. \tag{44}$$

Let us introduce the *reduced* $\mathbb{S}$ and $\mathbb{M}$ functions

$$\mathbb{S}_r := -\sum_{j\geqslant 1}\frac{1}{jz^j}S_{j+1}, \qquad \mathbb{M}_r := \frac{\partial\mathbb{S}_r}{\partial z}.$$

From (18) we see that

$$\mathbb{F}(\epsilon, x, \mathbf{t} - \epsilon[z^{-1}]) - \mathbb{F}(\epsilon, x, \mathbf{t}) = \epsilon\mathbb{S}_r(\epsilon, z, x, \mathbf{t}), \tag{45}$$

and by differentiating this equation with respect to z we obtain

$$\sum_{j\geqslant 1}\frac{1}{z^{j+1}}\frac{\partial}{\partial t_j}\mathbb{F}(\epsilon, x, \mathbf{t}) = \mathbb{M}_r(\epsilon, z, x, \mathbf{t} + \epsilon[z^{-1}]). \tag{46}$$

This identity can be rewritten in a more convenient form. Indeed (45) implies

$\mathbb{S}_r(\epsilon, z, x, \mathbf{t}) - \mathbb{S}_r(\epsilon, z, x, \mathbf{t} - \epsilon[z'^{-1}]) = \mathbb{S}_r(\epsilon, z', x, \mathbf{t}) - \mathbb{S}_r(\epsilon, z', x, \mathbf{t} - \epsilon[z^{-1}])$,

and by differentiating with respect to z and then taking the limit $z' \to z$ one finds

$$\mathbb{M}_r(\epsilon, z, x, \mathbf{t} - \epsilon[z^{-1}]) = \mathbb{M}_r(\epsilon, z, x, \mathbf{t}) + \epsilon \sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}(\epsilon, z, x, \mathbf{t} - \epsilon[z^{-1}]).$$

Thus (46) becomes

$$\sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{F}}{\partial t_j} = \mathbb{M}_r - \epsilon \sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}. \tag{47}$$

3.1. The Hermitian matrix model

Let us write the partition function of the Hermitian matrix model in terms of slow variables $\mathbf{t} := \epsilon \boldsymbol{t}$, where $\epsilon = 1/N$,

$$Z_n(N\mathbf{t}) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} (\mathrm{d}x_k e^{NV(x_k,\mathbf{t})})(\Delta(x_1, \ldots, x_n))^2, \qquad V(z, \mathbf{t}) := \sum_{k\geqslant 1} (t_k + \mathbf{c}_k) z^k. \tag{48}$$

The large-N limit of the model is determined by the asymptotic expansion of $Z_n(N\mathbf{t})$ for $n = N$ as $N \to \infty$,

$$Z_N(N\mathbf{t}) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} (\mathrm{d}x_k e^{NV(x_k,\mathbf{t})})(\Delta(x_1, \ldots, x_N))^2. \tag{49}$$

It is well known [3] that $Z_n(\boldsymbol{t})$ is a τ-function of the semi-infinite 1-Toda hierachy, then in view of (20) we may look for a τ-function $\tau(\epsilon, x, \mathbf{t})$ of the dispersionful 1-Toda hierarchy verifying

$$\tau(\epsilon, \epsilon n, \mathbf{t}) = Z_n(N\mathbf{t}), \tag{50}$$

and consequently

$$\tau(\epsilon, 1, \mathbf{t}) = Z_N(N\mathbf{t}). \tag{51}$$

We may express the $1/N$-expansions of the main objects of the Hermitian matrix model in terms of objects in the dispersionful 1-Toda hierarchy. For instance, from (47) the *one-loop correlator*

$$W(z) := \frac{1}{N} \sum_{j\geqslant 0} \frac{1}{z^{j+1}} \langle tr M^j \rangle = \frac{1}{z} + \frac{1}{N^2} \sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial \log Z_N(N\mathbf{t})}{\partial t_j}$$

becomes

$$W(z) = \frac{1}{z} + \mathbb{M}_r(\epsilon, z, 1, \mathbf{t}) - \epsilon \sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}(\epsilon, z, 1, \mathbf{t}). \tag{52}$$

Loop correlators of higher order can be obtained from $W(z)$ by application of the *loop-insertion* operator $\mathrm{d}/\mathrm{d}V(z)$ [6] ,

$$W(z_1, \ldots, z_s) = \frac{\mathrm{d}}{\mathrm{d}V(z_s)} \cdots \frac{\mathrm{d}}{\mathrm{d}V(z_2)} W(z_1)$$
$$\frac{\mathrm{d}}{\mathrm{d}V(z)} := \sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial}{\partial t_j}.$$

4. Semiclassical expansions

We now turn to the solutions of the system of string equations (23). The first equation is solved by setting

$$\mathcal{L} = \bar{\mathcal{L}} = \Lambda + u + v\Lambda^{-1},$$

which is in agreement with the asymptotic form (16) required for $\mathcal{L}$ and $\bar{\mathcal{L}}$.

Let us consider the second string equation of (23). We look for solutions $\mathcal{M}$ and $\overline{\mathcal{M}}$ verifying asymptotic expansions of the form (16). To this end, we first set

$$\mathcal{M} + F(\mathcal{L}) = \overline{\mathcal{M}} + \overline{F}(\overline{\mathcal{L}}) = \sum_{j=1}^{\infty} j(t_j + c_j)(\mathcal{L}^{j-1})_+ + \sum_{j=1}^{\infty} j(\bar{t}_j + \bar{c}_j)(\overline{\mathcal{L}}^{j-1})_-,$$

which, taking into account the first string equation, leads to

$$\begin{aligned} \mathcal{M} &= \sum_{j=1}^{\infty} j t_j \mathcal{L}^{j-1} + \sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))(\mathcal{L}^{j-1})_-, \\ \overline{\mathcal{M}} &= \sum_{j=1}^{\infty} j\bar{t}_j \mathcal{L}^{j-1} - \sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))(\mathcal{L}^{j-1})_+. \end{aligned} \tag{53}$$

In order to satisfy (16) and (22) we introduce auxiliary functions of the form

$$\begin{aligned} \Psi &= \exp \frac{1}{\epsilon}\left(\sum_{j=1}^{\infty} t_j z^j + x \log z - \sum_{j\geqslant 1} \frac{1}{j z^j} S_{j+1}\right), \\ \overline{\Psi} &= \exp \frac{1}{\epsilon}\left(\sum_{j=1}^{\infty} \bar{t}_j z^j - (x+\epsilon)\log z - \bar{S}_0 - \sum_{j\geqslant 1} \frac{1}{j z^j} \bar{S}_{j+1}\right), \end{aligned} \tag{54}$$

and impose

$$\begin{aligned} \mathcal{L}\Psi &= z\Psi, & \mathcal{M}\Psi &= \epsilon\frac{\partial \Psi}{\partial z}, \\ \bar{\mathcal{L}}\overline{\Psi} &= z\overline{\Psi}, & \overline{\mathcal{M}}\overline{\Psi} &= \epsilon\frac{\partial \overline{\Psi}}{\partial z}. \end{aligned} \tag{55}$$

Our aim is to determine u, v, $\mathcal{M}$ and $\overline{\mathcal{M}}$ from (55). Now, with the help of (41), we have that the equations (55) for the Orlov operators read

$$\begin{aligned} &\frac{x}{z} + \sum_{j\geqslant 2} \frac{1}{z^j} S_j = \sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))\left(\frac{1}{2z} T_{j[-1]} + \mathcal{O}\left(\frac{1}{z^2}\right)\right), \\ &-\frac{x+\epsilon}{z} + \sum_{j\geqslant 2} \frac{1}{z^j} \bar{S}_j = -\sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))\left(R_{j-1} + \frac{1}{2z} T_j + \mathcal{O}\left(\frac{1}{z^2}\right)\right). \end{aligned} \tag{56}$$

Matching the coefficients of z^{-1} in both sides of these two equations provides the same relation. Another relation is supplied by identifying the coefficients of the constant terms in the second equation of (56). Hence, we get a system of two equations to determine (u, v)

$$\begin{cases} \displaystyle\sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))R_{j-1} = 0, \\ \displaystyle\frac{1}{2}\sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))T_{j[-1]} = x. \end{cases} \tag{57}$$

By equating the coefficients of the remaining powers of z in (56) we characterize the functions S_j and $\overline{S}_j$ for $j \geqslant 1$ in terms of (u, v). Moreover, as it is proved below, the solution (u, v) provided by (57) is of the form

$$u = \sum_{k\geqslant 0} \epsilon^k u^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}), \qquad v = \sum_{k\geqslant 0} \epsilon^k v^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}),$$

with $v^{(2k+1)} = 0, \forall k \geqslant 0$. Thus, by solving (57) we characterize operators $(\mathcal{L}, \mathcal{M})$ and $(\bar{\mathcal{L}}, \overline{\mathcal{M}})$ which satisfy (23) and all the requirements of theorem 1. Therefore, they are solutions of the Lax equations for the dispersionful 2-Toda hierarchy.

We observe that, as it is noted by Takasaki and Takebe in [12], solving the system of string equations (23) does not determine the coefficient $\bar{S}_0$ in (54) and therefore it does not determine a wavefunction $\overline{\Psi}$ of the dispersionful 1-Toda hierarchy.

4.1. An iterative scheme for determining (u, v)

It is convenient to write (57) in the form

$$\oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i} z} U_z \mathbb{R}(z) = 0, \qquad \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} U_z \mathbb{T}(z) = -2(x+\epsilon), \tag{58}$$

where U denotes the function

$$U(z, \mathbf{t}, \bar{\mathbf{t}}) := \sum_{j=1}^{\infty} ((t_j + c_j) - (\bar{t}_j + \bar{c}_j)) z^j, \tag{59}$$

and γ is a large positively oriented closed path. Now by using the first identity of (36) and the two equations of (58) we find

$$\oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} U_z(\mathbb{T} + \mathbb{T}_{[-1]}) = \oint_\gamma \frac{\mathrm{d}z}{\pi \mathrm{i} z}(z-u) U_z \mathbb{R} = \oint_\gamma \frac{\mathrm{d}z}{\pi \mathrm{i}} U_z \mathbb{R} = -4x - 2\epsilon,$$

so that (57) reduces to a pair of equations involving $\mathbb{R}$ only

$$\begin{cases} \displaystyle\oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i} z} U_z(z) \mathbb{R}(z) = 0, \\ \displaystyle\oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} U_z(z) \mathbb{R}(z) = -2x - \epsilon. \end{cases} \tag{60}$$

These equations together with the system (36),

$$\begin{cases} \mathbb{T}_{[1]} + \mathbb{T} + \dfrac{2}{z}(u_{[1]} - z)\mathbb{R}_{[1]} = 0, \\ \mathbb{T}^2 - \dfrac{4}{z^2} v_{[1]} \mathbb{R}\mathbb{R}_{[1]} = 1, \end{cases} \tag{61}$$

give rise an iterative scheme for characterizing (u, v) as Taylor series in ϵ,

$$u = \sum_{k\geqslant 0} \epsilon^k u^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}), \qquad v = \sum_{k\geqslant 0} \epsilon^k v^{(k)}(x, \mathbf{t}, \bar{\mathbf{t}}).$$

The first step of the method is to determine the expansions

$$\mathbb{R}(z) = \sum_{k\geqslant 0} \epsilon^k R^{(k)}, \qquad \mathbb{T}(z) = \sum_{k\geqslant 0} \epsilon^k T^{(k)}, \tag{62}$$

in terms of (u, v). It can be done by equating the coefficients of powers of ϵ in (61). Indeed, the coefficients of ϵ^0 lead to

$$R^{(0)} = \frac{z}{((z-u^{(0)})^2 - 4v^{(0)})^{\frac{1}{2}}}, \qquad T^{(0)} = \frac{z - u^{(0)}}{((z-u^{(0)})^2 - 4v^{(0)})^{\frac{1}{2}}}, \tag{63}$$

and the coefficients of ϵ^l ($l \geqslant 1$) yield the following system:

$$T^{(l)} - (z - u^{(0)})\frac{R^{(l)}}{z} = \frac{1}{2}\sum_{\substack{i+j=l\\ j\geqslant 1}}\left(\frac{(-1)^j}{j!}\partial_x^j T^{(i)} + 2u^{(j)}\frac{R^{(i)}}{z}\right),$$

$$T^{(0)}T^{(l)} - 4v^{(0)}\frac{R^{(0)}}{z}\frac{R^{(l)}}{z} = 2\sum_{\substack{i+j+k=l\\ j<l}}\left(\sum_{i_1+i_2=i}\frac{1}{i_2!}\partial_x^{i_2}v^{(i_1)}\right)\frac{R^{(j)}}{z}\left(\sum_{\substack{k_1+k_2=k\\ k_1<l}}\frac{1}{k_2!}\partial_x^{k_2}\frac{R^{(k_1)}}{z}\right)$$
$$-\frac{1}{2}\sum_{\substack{i+j=l\\ i,j\geqslant 1}}T^{(i)}T^{(j)}. \qquad (64)$$

Some comments concerning these formulae are in order:

(i) Equations (64) determine each pair $(T^{(l)}, R^{(l)}/z)$ from $(T^{(j)}, R^{(j)}/z)$ with $j = 0, 1, \ldots, l-1$.

(ii) Equations (64) are linear with respect to $T^{(l)}, R^{(l)}/z$. Moreover, by taking into account (63), we see that the determinant of the coefficients of $T^{(l)}$ and $R^{(l)}/z$ in (64) is

$$[(z - u^{(0)})^2 - 4v^{(0)}]^{\frac{1}{2}}.$$

Hence it follows that the functions $R^{(l)}/z$ can be written as linear combinations of

$$\frac{z}{((z-u^{(0)})^2 - 4v^{(0)})^{r+\frac{1}{2}}}, \qquad \frac{1}{((z-u^{(0)})^2 - 4v^{(0)})^{r+\frac{1}{2}}}, \qquad r = 1, 2, \ldots, l+1$$

with coefficients depending on $u^{(j)}$ and $v^{(j)}$, with $j = 0, 1, \ldots, l$ and their x-derivatives only.

Now let us go back to the system (60) and find (u, v). By substituting the ϵ expansion of $\mathbb{R}$ into (60) we get a system of two equations for each $R^{(l)}$,

$$\begin{cases} \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i} z} U_z(z) R^{(l)}(z) = 0, \\ \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} U_z(z) R^{(l)}(z) = -2x\delta_{l0} - \delta_{l1}, \end{cases} \qquad (65)$$

which determine each pair $(u^{(l)}, v^{(l)})$ recursively. Furthermore, we can eliminate the explicit dependence on $(x, \mathbf{t}, \bar{\mathbf{t}})$ in the corresponding expressions since, by differentiating with respect to x the equations (65) for $l = 0$

$$\begin{cases} \frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{d}z \frac{U_z}{((z-u^{(0)})^2 - 4v^{[0]})^{\frac{1}{2}}} = 0, \\ \frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{d}z \frac{zU_z}{((z-u^{(0)})^2 - 4v^{(0)})^{\frac{1}{2}}} = -2x, \end{cases} \qquad (66)$$

all the integrals of the form

$$\frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{d}z \frac{U_z}{((z-u^{(0)})^2 - 4v^{(0)})^{r+\frac{1}{2}}}, \qquad \frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{d}z \frac{zU_z}{((z-u^{(0)})^2 - 4v^{(0)})^{r+\frac{1}{2}}}, \qquad (67)$$

can be expressed in terms of $(u^{(0)}, v^{(0)})$ and their x-derivatives. We observe that (67) are the variables introduced in [5] to determine the large N-expansion of the Hermitian matrix model.

Some important relations among the coefficients of the semiclassical expansions under consideration are found by realizing that given a solution $(u(\epsilon,x), v(\epsilon,x), \mathbb{R}(\epsilon,z,x), \mathbb{T}(\epsilon,z,x))$ of (60)–(61), then

$$\tilde{u}(\epsilon,x) := u(-\epsilon, x+\epsilon), \qquad \tilde{v}(\epsilon,x) := v(-\epsilon,x),$$
$$\tilde{\mathbb{R}}(\epsilon,z,x) := \mathbb{R}(-\epsilon,z,x+\epsilon), \qquad \tilde{\mathbb{T}}(\epsilon,z,x) := \mathbb{T}(-\epsilon,z,x+2\epsilon),$$

satisfies (60)–(61) as well. Thus, since the solution of (60)–(61) is uniquely determined by $(u^{(0)}, v^{(0)})$ we deduce that

$$\tilde{u}(\epsilon,x) = u(\epsilon,x), \qquad \tilde{v}(\epsilon,x) = v(\epsilon,x),$$
$$\tilde{\mathbb{R}}(\epsilon,z,x) = \mathbb{R}(\epsilon,z,x), \qquad \tilde{\mathbb{T}}(\epsilon,z,x) = \mathbb{T}(\epsilon,z,x).$$

Hence we find

$$\begin{aligned} u^{(2j-1)} &= \frac{1}{2}\sum_{k=1}^{2j-1}\frac{(-1)^{k+1}}{k!}\partial_x^k u^{(2j-1-k)}, \qquad v^{(2j-1)} = 0,\\ R^{(2j-1)} &= \frac{1}{2}\sum_{k=1}^{2j-1}\frac{(-1)^{k+1}}{k!}\partial_x^k R^{(2j-1-k)}, \qquad T^{(2j-1)} = \frac{1}{2}\sum_{k=1}^{2j-1}\frac{(-1)^{k+1}2^k}{k!}\partial_x^k T^{(2j-1-k)}, \end{aligned} \tag{68}$$

for $j = 1, 2, \ldots$.

4.1.1. Examples of calculations. Using (68) for $j = 1$, it is immediately found that

$$R^{(1)} = \frac{z\left((z-u^{(0)})u_x^{(0)} + 2v_x^{(0)}\right)}{2((z-u^{(0)})^2 - 4v^{(0)})^{\frac{3}{2}}}, \qquad T^{(1)} = \frac{4v^{(0)}u_x^{(0)} + 2(z-u^{(0)})v_x^{(0)}}{((z-u^{(0)})^2 - 4v^{(0)})^{\frac{3}{2}}}, \tag{69}$$

and

$$u^{(1)} = \tfrac{1}{2}u_x^{(0)}, \qquad v^{(1)} = 0. \tag{70}$$

With the help of *Mathematica*, one obtains

$$\begin{aligned} \frac{R^{(2)}}{z} &= \frac{4(z-u^{(0)})u^{(2)} + 8v^{(2)} + u_x^{(0)2} + 2v_{xx}^{(0)}}{4((z-u^{(0)})^2 - 4v^{(0)})^{\frac{3}{2}}}\\ &\quad + \frac{\frac{7}{2}v^{(0)}u_x^{(0)2} + \frac{5}{2}(z-u^{(0)})v_x^{(0)}u_x^{(0)} + (z-u^{(0)})v^{(0)}u_{xx}^{(0)} + 3v_x^{(0)2} + 2v^{(0)}v_{xx}^{(0)}}{((z-u^{(0)})^2 - 4v^{(0)})^{\frac{5}{2}}}\\ &\quad + \frac{10(z-u^{(0)})u_x^{(0)}v^{(0)}v_x^{(0)} + 10v^{(0)2}u_x^{(0)2} + 10v^{(0)}v_x^{02}}{((z-u^{(0)})^2 - 4v^{(0)})^{\frac{7}{2}}}, \end{aligned} \tag{71}$$

which leads to

$$\begin{aligned} u^{(2)} &= \frac{u_{xx}^{(0)}}{4} + \frac{v^{(0)}\left(7u_x^{(0)2}u_{xx}^{(0)} - 4u_{xx}^{(0)}v_{xx}^{(0)} - 2u_x^{(0)}v_{xxx}^{(0)}\right) + v_x^{(0)}\left(u_x^{(0)3} - 2u_x^{(0)}v_{xx}^{(0)} + 2v^{(0)}u_{xxx}^{(0)}\right)}{24\left(v_x^{(0)2} - v^{(0)}u_x^{(0)2}\right)}\\ &\quad + \frac{v^{(0)}u_x^{(0)}\left(u_x^{(0)4}v_x^{(0)} + 4v^{(0)}u_{xx}^{(0)}\left(u_x^{(0)3} - 2u_x^{(0)}v_{xx}^{(0)}\right) + 4v_x^{(0)}\left(v^{(0)}u_{xx}^{(0)2} + v_{xx}^{(0)2} - u_x^{(0)2}v_{xx}^{(0)}\right)\right)}{24\left(v_x^{(0)2} - v^{(0)}u_x^{(0)2}\right)^2}\\ v^{(2)} &= -\frac{v^{(0)2}u_x^{(0)}\left(u_x^{(0)5} + 4u_x^{(0)2}v_x^{(0)}u_{xx}^{(0)} - 4v_{xx}^{(0)}\left(u_x^{(0)3} + 2v_x^{(0)}u_{xx}^{(0)}\right) + 4u_x^{(0)}\left(v^{(0)}u_{xx}^{(0)2} + v_{xx}^{(0)2}\right)\right)}{24\left(v_x^{(0)2} - v^{(0)}u_x^{(0)2}\right)^2}\\ &\quad - \frac{v^{(0)}\left(u_x^{(0)4} - 3u_x^{(0)2}v_{xx}^{(0)} + 2u_x^{(0)}\left(2v_x^{(0)}u_{xx}^{(0)} + v^{(0)}u_{xxx}^{(0)}\right) + 2\left(v^{(0)}u_{xx}^{(0)2} + v_{xx}^{(0)2} - v_x^{(0)}v_{xxx}^{(0)}\right)\right)}{24\left(v_x^{(0)2} - v^{(0)}u_x^{(0)2}\right)} \end{aligned}$$

A further coefficient can be easily computed by taking $j = 2$ in (68). Thus we obtain

$$u^{(3)} = \tfrac{1}{2}u_x^{(2)} - \tfrac{1}{24}u_{xxx}^{(0)}, \qquad v^{(3)} = 0.$$

4.2. The classical limit

In the classical limit $\epsilon \to 0$ the functions (u, v) reduce to the first terms $(u^{(0)}, v^{(0)})$ of their semiclassical expansions and verify the equations of the dispersionless 1-Toda hierarchy

$$\partial_{t_j} u = \tfrac{1}{2}\partial_x(r_{j+1} - ur_j), \qquad \partial_{t_j} v = v\partial_x r_j, \tag{72}$$

where r_j are the coefficients of the Laurent expansion of $R := R^{(0)}$

$$R := \frac{z}{p - \bar{p}} = \frac{z}{\sqrt{(z-u)^2 - 4v}} = \sum_{k\geqslant 0} \frac{r_k(u, v)}{z^k}, \qquad r_0 = 1, \tag{73}$$

and we have taken into account (see (63)) that $\mathbb{T} = T^{(0)} = (z - u)R/z$. Here, $p := p^{(0)}$ and $\bar{p} := \bar{p}^{(0)}$ are given by

$$\begin{aligned} p(z) &= \tfrac{1}{2}\left((z-u) + \sqrt{(z-u)^2 - 4v}\right) = z - u - \frac{v}{z} + \cdots \\ \bar{p}(z) &= \tfrac{1}{2}\left((z-u) - \sqrt{(z-u)^2 - 4v}\right) = \frac{v}{z} + \cdots. \end{aligned} \tag{74}$$

According to (66) (u, v) are determined by

$$\begin{cases} \displaystyle\oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{U_z}{\sqrt{(z-u)^2 - 4v}} = 0, \\ \displaystyle\oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{zU_z}{\sqrt{(z-u)^2 - 4v}} = -2x, \end{cases} \tag{75}$$

which can also be expressed as *hodograph*-type equations

$$\begin{cases} \displaystyle\sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))r_{j-1} = 0, \\ \displaystyle\frac{1}{2}\sum_{j=1}^{\infty} j((\bar{t}_j + \bar{c}_j) - (t_j + c_j))r_j = x. \end{cases} \tag{76}$$

4.3. The planar limit of the Hermitian matrix model

From (52) the one-point correlator $W(z)$ is given by

$$W(z) = \frac{1}{z} + \mathbb{M}_r(\epsilon, z, 1, \mathbf{t}) - \epsilon \sum_{j\geqslant 1} \frac{1}{z^{j+1}} \frac{\partial \mathbb{S}_r}{\partial t_j}(\epsilon, z, 1, \mathbf{t}),$$

so that by using the first equations of (13) and (53) one finds

$$\begin{aligned} W(z) &= \sum_{j=0}^{\infty}(j+1)\left(\tilde{t}_{j+1} - \frac{\epsilon}{(j+1)z^{j+1}}\right)(\alpha_j + \beta_j p(z) - z^j) \\ &= -\sum_{j=1}^{\infty} j\left(\tilde{t}_j - \frac{\epsilon}{jz^j}\right)\left(-\frac{1}{2}R_{j-1} + \frac{z}{2\mathbb{R}}(z^{j-2}\mathbb{R})_\ominus - \left(\frac{z}{2\mathbb{R}}\mathbb{T}(z^{j-2}\mathbb{R})_\oplus\right)_\ominus\right), \end{aligned} \tag{77}$$

where

$$\tilde{t}_j := t_j + c_j.$$

We are going to show that the solution of the dispersionless 1-Toda hierarchy determined by (76) and $x = 1, \bar{t}_j = \bar{c}_j = 0$ describes the planar limit of the Hermitian matrix model in the *one-cut* case where the density of eigenvalues

$$\rho(z) = M(z)\sqrt{(z-a)(z-b)},$$

is supported on a single interval $[a, b]$. As it is known (see for instance [27, 28]) these objects are related to the first term $W^{(0)}$ of the large N-expansion of W in the form

$$W^{(0)} = -\frac{1}{2}V_z(z) + \mathrm{i}\pi\rho(z), \qquad V(z) := \sum_{k\geqslant 1}\tilde{t}_k z^k.$$

According to (63), in the classical limit $\mathbb{T} = T^{(0)} = (z-u)R/z$ and then from (77) it follows

$$W^{(0)} = \frac{1}{2}\sum_{j=1}^{\infty} j\tilde{t}_j r_{j-1} - \frac{1}{2}\sum_{j=1}^{\infty} j\tilde{t}_j z^{j-1} + \frac{1}{2}(p-\bar{p})\sum_{j=2}^{\infty} j\tilde{t}_j (z^{j-2}R)_{\oplus},$$

with $x = 1$ in all x-dependent functions. Due to the fact that $x = 1, \bar{t}_j = \bar{c}_j = 0$ the first hodograph equation (76) implies that the first term in the last equation vanishes. Therefore, the expressions for the density of eigenvalues and the end-points of its support provided the above solution of the dispersionless 1-Toda hierarchy are

$$\begin{aligned}&\rho(z) := \frac{1}{2\pi\mathrm{i}}\left(\frac{V_z}{\sqrt{(z-a)(z-b)}}\right)_{\oplus}\sqrt{(z-a)(z-b)},\\ &a := u - 2\sqrt{v}, \qquad b := u + 2\sqrt{v},\end{aligned} \tag{78}$$

where $x = 1$ in all x-dependent functions. Moreover, from (75), they are determined by the equations

$$\begin{cases}\displaystyle\oint_{\gamma}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{V_z}{\sqrt{(z-a)(z-b)}} = 0,\\ \displaystyle\oint_{\gamma}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{zV_z}{\sqrt{(z-a)(z-b)}} = -2.\end{cases} \tag{79}$$

They coincide with the equations for the planar limit contribution to the partition function of the Hermitian model [24–28].

5. Critical points and the double scaling limit

As we have seen the characterization of (u, v) as semiclassical expansions relies on the determination of smooth leading terms $(u^{(0)}, v^{(0)})$, which are defined implicitly by the hodograph equations (66). However, near critical points the functions $(u^{(0)}, v^{(0)})$ are multivalued and have singular x-derivatives. Thus the semiclassical expansions are not longer valid and a different procedure must be used. In this subsection, we indicate how the so-called *double scaling limit* method (see for instance [29]) can be formulated in our scheme.

To simplify the discussion we set $u \equiv 0$ and

$$t_{2j-1} = c_j = 0, \qquad j \geqslant 1; \qquad \bar{t}_j = \bar{c}_j = 0, \qquad j \geqslant 1, \tag{80}$$

so that the Lax operator is of the form

$$\mathcal{L} = \Lambda + v\Lambda^{-1}, \tag{81}$$

and we are only considering the Toda flows associated with the even times t_{2j}. After eliminating $\mathbb{R}$ in (36), one sees that the generating function $\mathbb{U} := \mathbb{T}_{[-1]}$ satisfies the identity

$$v(\mathbb{U} + \mathbb{U}_{[-1]})(\mathbb{U} + \mathbb{U}_{[1]}) = z^2(\mathbb{U}^2 - 1). \tag{82}$$

This leads to expansions of the form

$$\mathbb{U}=\sum_{j\geqslant 0}\frac{U_{2j}}{z^{2j}},\qquad \mathbb{U}=\sum_{k\geqslant 0}\epsilon^{2k}U^{(k)}. \tag{83}$$

On the other hand, the system (57) reduces to

$$-\sum_{j=1}^{\infty}jt_{2j}U_{2j}=-\frac{1}{4\pi\mathrm{i}}\oint_{\gamma}\mathrm{d}z V_z\mathbb{U}=x, \tag{84}$$

where $V=\sum_{k\geqslant 1}t_{2k}z^{2k}$. Thus, the solution v is found from (82) and (84). In particular, the leading term $v^{(0)}$ is implicitly determined by the hodograph equation

$$H(\boldsymbol{t}_{\text{even}},v^{(0)})=x, \tag{85}$$

where

$$H(\boldsymbol{t}_{\text{even}},v):=-\frac{1}{4\pi\mathrm{i}}\oint_{\gamma}\mathrm{d}z\, V_zU^{(0)}=-\frac{1}{4\pi\mathrm{i}}\oint_{\gamma}\mathrm{d}z\frac{zV_z}{(z^2-4v)^{\frac{1}{2}}}.$$

Given a general mth order critical point $v_c:=v_c(\boldsymbol{t}_{\text{even}})$ satisfying

$$\left.\frac{\partial H}{\partial v}\right|_{v_c}=\cdots=\left.\frac{\partial^{m-1}H}{\partial v^{m-1}}\right|_{v_c}=0,\qquad \left.\frac{\partial^m H}{\partial v^m}\right|_{v_c}\neq 0,$$

the method of the double scaling limit introduces a new small parameter $\tilde{\epsilon}$ and a new variable $\tilde{x}$ given by

$$\tilde{\epsilon}:=\epsilon^{\frac{2}{2m+1}},\qquad x=H(v_c)+\tilde{\epsilon}^m\tilde{x}, \tag{86}$$

and generates solutions to (82) and (84) of the form

$$v=v_c\left(1+\sum_{k\geqslant 1}\tilde{\epsilon}^k u^{(k)}\right),\qquad \mathbb{U}=\sum_{k\geqslant 0}\tilde{\epsilon}^k\tilde{U}^{(k)}. \tag{87}$$

To prove it, we first observe that $\epsilon\partial_x=\tilde{\epsilon}^{1/2}\partial_{\tilde{x}}$, so that (82) can be rewritten as

$$v\sum_{n\geqslant 1}\tilde{\epsilon}^n\left(\frac{4}{(2n)!}\mathbb{U}\partial_x^{2n}\mathbb{U}+\sum_{k+l=2n;k,l\geqslant 1}\frac{(-1)^k}{k!l!}\partial_x^k\mathbb{U}\partial_x^l\mathbb{U}\right)=(z^2-4v)\mathbb{U}^2-z^2, \tag{88}$$

and by substituting the expansions (87) in this identity and equating $\tilde{\epsilon}$-powers one can express each coefficient $\tilde{U}^{(n)}$ in the form

$$\tilde{U}^{(n)}=\sum_{r=1}^{n}\frac{zv_c^rG_{n,r}}{(z^2-4v_c)^{\frac{2r+1}{2}}}, \tag{89}$$

where the coefficients $G_{n,r}$ are differential polynomials in $u^{(k)}$, $1\leqslant k\leqslant n-r+1$ and their $\tilde{x}$-derivatives. In particular

$$G_{n,1}=2u^{(n)},$$

and the first few $\tilde{U}^{(n)}$ are

$$\tilde{U}^{(0)} = \frac{z}{(z^2-4v_c)^{\frac{1}{2}}}, \qquad \tilde{U}^{(1)} = \frac{2v_c z u^{(1)}}{(z^2-4v_c)^{\frac{3}{2}}},$$

$$\tilde{U}^{(2)} = \frac{2v_c z u^{(2)}}{(z^2-4v_c)^{\frac{3}{2}}} + \frac{2{v_c}^2 z\left(3{u^{(1)}}^2+\partial_{\tilde{x}}^2 u^{(1)}\right)}{(z^2-4v_c)^{\frac{5}{2}}},$$

$$\tilde{U}^{(3)} = \frac{2v_c z u^{(3)}}{(z^2-4v_c)^{\frac{3}{2}}} + \frac{{v_c}^2 z\left(12u^{(1)}\left(6u^{(2)}+\partial_{\tilde{x}}^2 u^{(1)}\right)+12\partial_{\tilde{x}}^2 u^{(2)}+\partial_{\tilde{x}}^4 u^{(1)}\right)}{6(z^2-4v_c)^{\frac{5}{2}}} + \frac{2{v_c}^3 z\left(10(u^{(1)})^3+5(\partial_{\tilde{x}} u^{(1)})^2+10u^{(1)}\partial_{\tilde{x}}^2 u^{(1)}+\partial_{\tilde{x}}^4 u^{(1)}\right)}{(z^2-4v_c)^{\frac{7}{2}}}.$$

Note that $\tilde{U}^{(0)}(v) = U^{(0)}(v)$.

By substituting (86)–(87) into (84) we get the system

$$\begin{cases} \oint_\gamma \mathrm{d}z\, V_z \tilde{U}^{(j)} = 0, & j = 1,\ldots,m-1, \\ -\dfrac{1}{4\pi\mathrm{i}}\oint_\gamma \mathrm{d}z\, V_z \tilde{U}^{(n)} = \delta_{nm}\tilde{x}, & n \geqslant m. \end{cases} \tag{90}$$

Since v_c is a mth order critical point of (85) we have that

$$\oint_\gamma \mathrm{d}z \frac{V_z}{(z^2-4v_c)^{\frac{2j+1}{2}}} = 0, \qquad j = 1,\ldots,m-1.$$

Hence, in view of (89), the first $(m-1)$ equations in (90) are identically satisfied while the remaining ones become

$$-\sum_{r=m}^{n} \frac{v_c^r G_{n,r}}{4\pi\mathrm{i}} \oint_\gamma \mathrm{d}z \frac{zV_z}{(z^2-4v_c)^{\frac{2r+1}{2}}} = \delta_{nm}\tilde{x}, \qquad n \geqslant m. \tag{91}$$

For $n = m$, we get the equation which determines the leading contribution $u^{(1)}$ in the double scaling limit

$$G_{m,m}(u^{(1)}) = K_m\tilde{x}, \qquad K_m^{-1} := {v_c}^m \oint \frac{\mathrm{d}z}{4\pi\mathrm{i}} \frac{V_z z}{(z^2-4v_c)^{\frac{2m+1}{2}}}. \tag{92}$$

For example

$$m = 2, \qquad 2\left(3{u^{(1)}}^2+\partial_{\tilde{x}}^2 u^{(1)}\right) = K_2\tilde{x};$$

$$m = 3, \qquad 2\left(10(u^{(1)})^3+5(\partial_{\tilde{x}} u^{(1)})^2+10u^{(1)}\partial_{\tilde{x}}^2 u^{(1)}+\partial_{\tilde{x}}^4 u^{(1)}\right) = K_3\tilde{x}$$

For $n \geqslant m+1$, the equations of the system (91) characterize the coefficients $u^{(k)}$ for $k \geqslant 2$.

The differential equations (92) for $u^{(1)}$ are essentially the stationary KdV equations [30]. Indeed, from (88) and taking into account (89) one gets ($G_i := G_{i,i}$, $G_i' := \partial_{\tilde{x}} G_i, \ldots$)

$$2v_c \sum_{i+j=m-1} G_i G_j'' - \sum_{i+j=m} G_i G_j + 4v_c u^{(1)} \sum_{i+j=m-1} G_i G_j - v_c \sum_{i+j=m-1} G_i' G_j' = 0,$$

which, up to trivial rescalings, coincides with the equation verified by the coefficients of the expansion of the resolvent diagonal R of the Schödinger operator

$$RR'' - 2(z^2-u)R^2 - \frac{1}{2}R'^2 + 2z^2 = 0, \qquad R = 1 + \sum_{j\geqslant 1} \frac{R_j}{z^{2j}}.$$

6. Conclusion and outlook

We have seen how to determine solutions of a relevant class of string equations of the dispersionful Toda hierarchy by using standard techniques of the theory of integrable systems. They are characterized by means of a well-defined iterative scheme and expressed in terms of semiclassical series for the coefficients (u, v) of the Lax operator (9). We also saw that one of these solutions underlies the large-N limit of the Hermitian matrix model. In contrast to the traditional approach based on the use of the *discrete* linear system (2) and the Lax–Orlov operators (5), this special solution is directly provided from our iterative scheme without any further continuum limit operation. This is the main advantage of our approach and it is a consequence of the fact that our starting point, the dispersionful Toda hierarchy, already encodes a continuum limit of the standard Toda hierarchy. Another advantage is that a simple formulation of the double scaling limit for regularizing the semiclassical series at critical points can be derived.

Our method can be also applied to the study of the large-N limit of the normal matrix model which is related to string equations of the Toda hierarchy [31–38] of the form

$$\bar{\mathcal{L}} = \mathcal{M}, \qquad \overline{\mathcal{M}} = -\mathcal{L}. \tag{93}$$

The dispersionless limit of these string equations is deeply related to interesting integrable models of Laplacian growth processes and Hele–Shaw problems with zero surface tension (see [17, 18] for some methods of solution). These problems are ill-posed since an initially smooth interface often becomes singular in the process of evolution and cusp-like singularities occur at finite time. They correspond to the critical points of the corresponding solutions of the dispersionless Toda hierarchy. On the other hand, recent works [39–41] propose to regularize some of these critical regimes by means of solutions of dispersionful hierarchies like the KdV and AKNS hierarchies. In particular, it should be noted that the hodograph equations of the dispersionless AKNS hierarchy (see [41], equations (29)–(33)) used to describe the bubble break-off in Hele–Shaw flows coincide with the hodograph equations (76) of the dispersionless Toda hierarchy for $\bar{t}_j = \bar{c}_j = c_j = 0, \forall j$. Thus it may expected that matching methods based on the double-scaling limit of dispersionful versions of the Toda, KdV and AKNS hierarchies will play a fundamental role in the nearest future.

The generalization of the methods of the present work to study the Hermitian matrix model beyond the one-cut case is an open problem. As it was proved in [42], the coefficients (u_n, v_n) of the Lax operator

$$\mathcal{L} = \Lambda + u_n + v_n \Lambda^T,$$

of the standard Toda Hierarchy in the multi-cut case are quasi-periodic in n, so that there is no regular large-N expansion involving only power series in $1/N$. Therefore, the dispersionful Toda hierarchy used in this paper, which is based on expansions of the form (10), is no longer appropriate and other candidates to describe the continuum limits of the Toda hierarchy must be considered.

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